Sankhyā : The Indian Journal of Statistics
1978, Volume **39**, Series B, Pt. 4, pp. 344–361.

RANKING AND SELECTION PROBLEMS FOR NORMAL POPULATIONS WITH COMMON KNOWN COEFFICIENT OF VARIATION

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SUMMARY. The problem of selecting the normal population having the largest mean has been widely studied in the literature by treating the population means and the variances as unrelated parameters. But it is very common in practice to find that the population standard deviations are related to the means by a proportionality relation ; the constant of proportionality being known as the coefficient of variation. In this paper single-stage ranking and selection procedures are proposed for the above situation where the populations under study have a common known coefficient of variation. The indifference-zone approach and the subset selection approach are both considered. The large sample theory is studied in detail and the corresponding tables are provided for implementing the proposed procedures. The small sample theory is discussed briefly in the Appendix.

1. Introduction

The problem of choosing the population having the largest mean where the populations under study are assumed to be normal is a very important one in practice and has received most attention in the ranking and selection literature. In his pioneering paper, Bechhofer (1954) proposed the so-called indifference-zone approach to this problem (referred to as the normal means problem hereafter) and gave a single-stage procedure for the case of known variances. For the case of unknown variances two-stage procedures were studied by Bechhofer, Dunnett and Sobel (1954) and Dudewicz and Dalal (1975). Another so-called subset selection approach to the normal means problem was proposed by Gupta (1956 and 1965) who gave single-stage procedures for the cases of common known variance and common unknown variance (see also Gupta and Sobel, 1957). Recent reviews of the literature on the normal means problem may be found in Wetherill and Ofosu (1974) and Bechhofer (1975).

All of the work on the normal means problem so far assumed that the population means and the variances are *unrelated* parameters. However, in practice, as Gleser and Healy (1976) recently pointed out, it is common to find, particularly in physical and biological applications, that the population standard deviations are proportional to their means; the constant of proportionality being known as the coefficient of variation. In such a situation

where the population means and the variances may be assumed to be *related* parameters, the procedures those have been developed for the case of unknown variances would be inappropriate. The knowledge of the proportionality relation between the population means and the standard deviations should be used to develop new ranking and selection procedures. This is the purpose of the present paper.

In particular we propose single-stage procedures (for two different goals) which use estimators developed by Gleser and Healy (1976) for the case where the coefficient of variation is known to the experimenter. This would be a reasonable assumption in practice if the data from past experiments dealing with similar populations (e.g., yield data about similar varieties of a grain from the records of previous field trials) are available which would enable a fairly accurate estimate of the coefficient of variation to be obtained from the scatterplots of the sample means and the sample standard deviations. The case of unknown coefficient of variation cannot be handled by a single-stage procedure (at least for the goal employing the indifference-zone approach); this problem is under study at present.

The statistical formulation of the problem is given in Section 2. In Section 3 we discuss the choice of the estimator for the means to be used in our procedures. We then point out certain difficulties associated with the practical implementation of the exact small sample results for the selected estimators and how these difficulties may be overcome using the large sample theory. In Section 4 we give large sample results for the procedures which use any best asymptotically normal (B.A.N) estimator for the means. The adequacy of the large sample approximation is investigated by a simulation study and it is found that the large sample approximation is quite good but little bit on the deficient side. The tables for implementing the procedures in the large sample case are given at the end of the paper. These tables are related to the tables of the equi-correlated multivariate normal distribution published by Gupta, Nagel and Panchapakesan (1973) and, therefore. they have additional applications as discussed by these authors. Finally we give some concluding remarks in Section 5.

2. Preliminaries

2.1. *Assumptions and notation.* Let π_i $(1 \leqslant i \leqslant k)$ denote a normal population with unknown mean θ_i and variance $\alpha\theta_i^2$ where $\alpha > 0$ is assumed to be known. As explained in Gleser and Healy (1976) we shall assume, without loss of generality, that the coefficient of variation $\sqrt{\alpha}$ and the means $\theta_i (1 \leqslant i \leqslant k)$ are positive constants. Let $\theta_{[1]} \leqslant \theta_{[2]} \leqslant \ldots \leqslant \theta_{[k]}$ be the

ordered values of the means. We shall assume that the experimenter has no prior knowledge concerning the correct pairing between π_i and $\theta_{[j]}$ $(1 \leqslant i, j \leqslant k)$ and the population corresponding to $\theta_{[k]}$ (assumed to be unique) will be referred to as the "best" population. In the following we shall use Ω to denote the *parameter space* and $\boldsymbol{\theta}$ to denote the parameter vector $(\theta_1, \ldots, \theta_k)'$.

2.2. *Indifference-zone approach.* According to this approach the *goal* (*Goal* I) of the experimenter is to select the best population. The selection of the best population is referred to as the *correct selection* (CS).

To specify the *probability requirement* according to this approach, it is first necessary to define a "distance function" between any two populations. Since θ_i is a scale parameter in the distribution of the observations from π_i $(1 \leqslant i \leqslant k)$ we shall define the distance between $\theta_{[i]}$ and $\theta_{[j]}$ as their ratio $\delta_{ij} = \theta_{[i]}/\theta_{[j]}$ $(1 \leqslant i, j \leqslant k)$. The experimenter restricts consideration to only those procedures which guarantee the probability requirement that

$$P_{\boldsymbol{\theta}}(\mathrm{CS}) \geqslant P^* \forall \boldsymbol{\theta} \in \Omega(\delta^*) \qquad \ldots \quad (2.1)$$

where $\delta^* > 1$ and $1/k < P^* < 1$ are preassigned constants. The subset $\Omega(\delta^*) = \{\boldsymbol{\theta} \epsilon \Omega \mid \delta_{k,k-1} \geqslant \delta^*\}$ is known as the *preference-zone*; the *indifference-zone* being its complement in Ω.

2.3. *Subset selection approach.* According to this approach the goal (*Goal* II) of the experimenter is to select a (preferably small but non-empty) subset of populations which contains the best population. The selection of *any* subset containing the best population is referred to as a *correct selection* (CS).

It is not necessary to define a distance function in order to specify the *probability requirement* in this approach. The experimenter restricts consideration to only those procedures which guarantee the probability requirement that

$$P_{\boldsymbol{\theta}}(\mathrm{CS}) \geqslant P^* \forall \boldsymbol{\theta} \in \Omega, \qquad \ldots \quad (2.2)$$

where P^*, $1/k < P^* < 1$, is a preassigned constant.

3. Choice of the estimator for the means

The procedures for Goals I and II would be based on some estimator for the means θ_i $(1 \leqslant i \leqslant k)$. A number of estimators of the mean of the normal distribution when the coefficient of variation is known have been proposed in the literature and from these we must make a choice of the estimator to be used in our procedures. First we make a brief review of some of these estimators.

Consider a random sample $X_1, X_2, \ldots, X_n$ of fixed size $n \geqslant 2$ taken from $N(\theta, \alpha\theta^2)$ where $\theta > 0$ and $\sqrt{\alpha} > 0$ is known. Let $\bar{X} = \sum_{j=1}^{n} X_j/n$ and $S^2 = \sum_{j=1}^{n} (X_j - \bar{X})^2/(n-1)$ be the sample mean and the sample variance respectively. Gleser and Healy (1976) have discussed the following estimators of θ: the minimum variance estimator (T_{LU}) in the class of all unbiased estimators which are linear in $\bar{X}$ and S (Khan 1968), the minimum mean squared error (MSE) estimator (T_{LMMS}) in the class of all estimators which are linear in $\bar{X}$ and S, the positive part (T^+_{LMMS}) of T_{LMMS}, the maximum likelihood estimator (T_{MLE}), and the minimum MSE scale equivariant estimator (T_{MMSS}). All these estimators are B.A.N. and the asymptotic distribution of $\sqrt{n}(T-\theta)$, where T denotes any B.A.N. estimator of θ, is $N(0, \alpha\theta^2/(1+2\alpha))$. It should be noted that $\bar{X}$ is an inadmissible estimator of θ and hence should not be used.

Out of these estimators, T^+_{LMMS} and T_{MMSS} are the only ones which besides having the optimal properties mentioned above are almost surely (a.s.) non-negative. This latter property is a desirable one for an estimator to have for its use particularly in the procedure for Goal II; see Section 4.2. Therefore we shall restrict our choice to only these two estimators whose expressions are given below. First we define constants

$$a_n = \{(n-1)/2\alpha\}^{\frac{1}{2}}\Gamma\{(n-1)/2\}/\Gamma(n/2),$$

$$b_n = \alpha a_n^2 - 1,$$

$$c_n = a_n\alpha/\{\alpha+(\alpha+n)b_n\},$$

$$d_n = nb_n/\{\alpha+(\alpha+n)b_n\}.$$

Then following are the expressions for T^+_{LMMS} and T_{MMSS}:

$$T^+_{LMMS} = \begin{cases} c_nS+d_n\bar{X} & \text{if } c_nS+d_n\bar{X} > 0 \\ 0 & \text{if } c_nS+d_n\bar{X} \leqslant 0, \end{cases}$$

and

$$T_{MMSS} = \left(\alpha^{-1}\sum_{j=1}^{n} X_j^2\right)^{\frac{1}{2}} \left\{\int_0^\infty u^n e^{-0.5u^2+Wu}du\right\} \Big/ \left\{\int_0^\infty u^{n+1}e^{-0.5u^2+Wu}du\right\},$$

where

$$W = \left(\sum_{j=1}^{n} X_j\right) \Big/ \left(\alpha\sum_{j=1}^{n} X_j^2\right)^{\frac{1}{2}}.$$

From a computational viewpoint, it appears that T^{+}_{LMMS} should be preferred. However we should note that Gleser and Healy (1976) have given a method to evaluate T_{MMSS} by means of continued fractions which simplifies its computation considerably. More importantly T_{MMSS} has uniformly smaller MSE in comparison to T^{+}_{LMMS} and, therefore, it is to be preferred.

It is not possible to obtain the exact sampling distribution of T_{MMSS} in a closed form whereas that of T^{+}_{LMMS} can be easily obtained. But even in the case of T^{+}_{LMMS}, practical implementation of the small sample distribution theory as applied to procedures for Goal I and Goal II problems is inhibited because of several difficulties. These difficulties can be best illustrated in the context of Goal I. For this goal, the statistician's task is to provide a sampling and decision procedure (which we shall take to be a single-stage procedure) and the associated (common) size of the sample to be observed from each population which would guarantee (2.1). The necessary sample size can be computed based on the exact small sample distribution theory. These sample sizes will have to be tabulated for each combination of values of k, δ^*, P^* and α. Therefore the corresponding tables would be relatively unwieldy. Secondly, the discrete search for the suitable value of the sample size n necessary to guarantee (2.1) for any specified values of k, δ^*, P^* and α would be computationally very costly.

These difficulties can be obviated to a significant extent by employing the large sample distribution theory. Thus in the large sample theory, by a suitable parametrization of n and α, it becomes necessary to make the tabulations only for each combination of values of k, δ^* and P^*. Therefore the corresponding tables are relatively compact. Also the problem of discrete search is replaced by that of solving a certain integral equation which can be done quite economically on a digital computer. Finally the large sample results are applicable to procedures based on any B.A.N. estimator for the population means. Because of these advantages, we concentrate mainly on the large sample theory in the present paper. The small sample theory is given in the Appendix for the interested reader.

We wish to point out here that an apparent discrepancy arises when the large sample results are used as an approximation in the case of small samples for procedures employing T_{MMSS} as the estimator for the means. This discrepancy can be best explained in the context of Goal II (subset selection approach). For this goal, we can use the natural selection procedure (procedure $\mathscr{P}_{II}$ given in Section 4.2) for scale parameter families proposed by Gupta (1965) which employs T_{MMSS} as the estimator for the means. It is clear that

such a procedure *exists* for any P^*-value for any fixed sample size n. (That is, such a procedure guarantees (2.2) for any P^*-value for any fixed n). However, using the large sample results it can be shown that such a procedure *does not exist* for P^*-values larger than a certain upper bound < 1 unless the sample size n is made arbitrarily large. This discrepancy is relatively minor from a practical viewpoint, since if $n \geqslant 6$ then the upper bound is extremely close to 1 as we show in Section 4.2.1.

It can be shown using the exact small sample results that the natural selection procedure employing T^+_{LMMS} as the estimator for the means does not exist for large values of P^* unless the sample size n is made arbitrarily large; see the Appendix for the details. This last fact may be regarded as an additional reason for preferring the use of T_{MMSS} as the estimator for the means in the procedures to be discussed below.

For later reference we note here that the distributions of T^+_{LMMS} and T_{MMSS} are stochastically increasing (SI) in θ. This follows since T^+_{LMMS} and T_{MMSS} are a.s. nonnegative and scale equivariant estimators of θ (i.e., for $c > 0$, $T(cX_1, \ldots, cX_n) = cT(X_1, \ldots, X_n)$). Furthermore, the distribution of X_j/θ $(1 \leqslant j \leqslant n)$ is independent of θ. Therefore, θ is a scale parameter in the distributions of T^+_{LMMS} and T_{MMSS} which are thus SI in θ.

4. Large sample results

4.1. *Procedure for Goal* I (*indifference-zone approach*). We propose the following single-stage procedure $\mathscr{P}_I$ for Goal I: Take $n \geqslant 2$ independent observations from each π_i, compute a B.A.N. estimator T_i of θ_i $(1 \leqslant i \leqslant k)$ and decide that the population associated with $T_{\max} = \max\limits_{1 \leqslant i \leqslant k} T_i$ is best. Although the possibility of ties can be ignored in the large sample probability calculations, in practice ties will occur (with nonzero probability for T^+_{LMMS} in the case of small samples; see the Appendix) in which case one of the populations tying for $T_{\max}$ may be selected at random.

4.1.1. *Probability of correct selection of* $\mathscr{P}_I$ *and its infimum.* To find the minimum sample size n necessary so that (2.1) is guaranteed using $\mathscr{P}_I$ we first derive the asymptotic expression for $P_{\boldsymbol{\theta}}(\mathrm{CS})$ of $\mathscr{P}_I$ and then find its infimum over $\Omega(\delta^*)$. This is done in the following theorem.

Theorem 4.1: *For* $\mathscr{P}_I$ *we have that for any* $\boldsymbol{\theta} \in \Omega$

$$P_{\boldsymbol{\theta}}(CS) = \cong \int_{-\infty}^{\infty} \prod_{i=1}^{k-1} \Phi\left[t\delta_{ki} + (\delta_{ki} - 1)\lambda\right] \varphi(t)\, dt \qquad \ldots \quad (4.1)$$

where $\Phi(\cdot)$ *denotes the standard normal c.d.f,* $\varphi(\cdot)$ *the corresponding p.d.f and* $\lambda = \{n(1+2\alpha)/\alpha\}^{\frac{1}{2}}$. *The infimum of* $P_{\boldsymbol{\theta}}(CS)$ *over* $\Omega(\delta^*)$ *is achieved for any* $\boldsymbol{\theta}$ *such that* $\delta_{ki} = \delta^* \; \forall \; i \neq k$ (*known as the least -favourable or LF-configuration*) *and*

$$\inf_{\Omega(\delta^*)} P_{\boldsymbol{\theta}}(CS) = \int_{-\infty}^{\infty} \Phi^{k-1}[t\delta^* + (\delta^* - 1)\lambda]\, \varphi(t)\, dt. \quad \ldots \quad (4.2)$$

Proof: We denote by $T_{(i)}$ the B.A.N. estimator corresponding to $\theta_{[i]}$ $(1 \leqslant i \leqslant k)$. We have that

$$\begin{aligned} P_{\boldsymbol{\theta}}(\mathrm{CS}) &= P_{\boldsymbol{\theta}}\{T_{(k)} > T_{(i)} \forall \; i \neq k\} \\ &= P_{\boldsymbol{\theta}}\{[(T_{(k)} - \theta_{[k]})/\theta_{[k]}\lambda^{-1}]\, \delta_{ki} + (\delta_{ki} - 1)\lambda \\ &\qquad > [(T_{(i)} - \theta_{[i]})/\theta_{[i]}\lambda^{-1}] \forall i \neq k\} \end{aligned}$$

which yields (4.1).

The proof of the infimum of $P_{\boldsymbol{\theta}}(\mathrm{CS})$ is somewhat tedious. (For T^{+}_{LMMS} and T_{MMSS} the proofs would be straightforward since their distributions are SI in θ.) Consider the parameter configuration $\theta_{[1]} = \ldots = \theta_{[l]} = \theta < \theta_{[l+1]}$ for some l, $1 \leqslant l \leqslant k-1$ and denote the corresponding $P_{\boldsymbol{\theta}}(\mathrm{CS})$ by $Q(\theta)$. Let $\delta_k = \theta_{[k]}/\theta$. Then we have

$$Q(\theta) = \int_{-\infty}^{\infty} \Phi^{l}[t\delta_k + (\delta_k - 1)\lambda] \prod_{i=l+1}^{k-1} \Phi[t\delta_{ki} + (\delta_{ki} - 1)\lambda]\, \varphi(t)\, dt.$$

Hence

$$\frac{dQ}{d\theta} = -\frac{l\theta_{[k]}}{\theta^2}\{I_1 + \lambda I_2\}, \quad \ldots \quad (4.3)$$

where

$$I_1 = \int_{-\infty}^{\infty} t\, \Phi^{l-1}[t\delta_k + (\delta_k - 1)\lambda] \prod_{i=l+1}^{k-1} \Phi[t\delta_{ki} + (\delta_{ki} - 1)\lambda]\, \varphi[t\delta_k + (\delta_k - 1)\lambda]\, \varphi(t)\, dt$$

and

$$I_2 = \int_{-\infty}^{\infty} \Phi^{l-1}[t\delta_k + (\delta_k - 1)\lambda] \prod_{i=l+1}^{k-1} \Phi[t\delta_{ki} + (\delta_{ki} - 1)\lambda]\, \varphi[t\delta_k + (\delta_k - 1)\lambda]\, \varphi(t)\, dt.$$

In I_1 integrate by parts by regarding $t\varphi(t)\,dt = dv$ to obtain

$$
\begin{aligned}
I_1 = & -\delta_k^2 I_1 - \delta_k(\delta_k - 1)\lambda I_2 \\
& + (l-1)\delta_k \int_{-\infty}^{\infty} \Phi^{l-2}[t\delta_k + (\delta_k - 1)\lambda] \\
& \times \prod_{i=l+1}^{k-1} \Phi[t\delta_{ki} + (\delta_{ki} - 1)\lambda]\varphi^2[t\delta_{ki} + (\delta_{ki} - 1)\lambda]\,\varphi(t)\,dt \\
& + \sum_{j=l+1}^{k-1} \delta_{kj} \int_{-\infty}^{\infty} \Phi^{l-1}[t\delta_k + (\delta_k - 1)\lambda] \\
& \times \prod_{\substack{i=l+1 \\ i\neq j}}^{k-1} \Phi[t\delta_{ki} + (\delta_{ki} - 1)\lambda]\,\varphi[t\delta_{kj} + (\delta_{kj} - 1)\lambda] \\
& \times \varphi[t\delta_k + (\delta_k - 1)\lambda]\,\varphi(t)\,dt \\
= & -\delta_k^2 I_1 - \delta_k(\delta_k - 1)\lambda I_2 + J_1 + J_2 \text{ (say).} \qquad \ldots \quad (4.4)
\end{aligned}
$$

Solving for I_1 from (4.4) and substituting in (4.3) we obtain

$$
\frac{dQ}{d\theta} = -\frac{l\theta_{[k]}}{\theta^2(\delta_k^2 + 1)}\{(\delta_k + 1)\lambda I_2 + J_1 + J_2\} \leqslant 0
$$

and therefore $Q(\theta)$ is nonincreasing in θ. Since this is true for each l, $1 \leqslant l \leqslant k-1$, it is clear that the infimum of $P_{\boldsymbol{\theta}}$ (CS) is achieved by increasing each $\theta_{[i]}$ $(1 \leqslant i \leqslant k-1)$ so that $\delta_{ki} = \delta^* \forall\, i \neq k$. The expression (4.2) is now obvious.

Let $\lambda(k, \delta^*, P^*)$ denote the solution in λ to the equation obtained by setting the r.h.s. of (4.2) equal to P^*. Then the probability requirement (2.1) can be guaranteed by choosing n to be the smallest integer $\geqslant \alpha\lambda^2(k, \delta^*, P^*)/(1+2\alpha)$ in procedure $\mathscr{P}_I$. Since the r.h.s. of (4.2) is strictly increasing in λ for $\delta^* > 1$, a unique solution exists for each P^*-value for $1/k < P^* < 1$.

It may be of some interest to note that for the so-called generalized LF-configuration (i.e., $\delta_{ki} = \delta > 1 \forall\, i \neq k$) for fixed $\lambda > 0$ if $\delta \to 1$ then $P(\text{CS}) \to 1/k$ and if $\delta \to \infty$ then $P(\text{CS}) \to \Phi(\lambda) < 1$. This latter result is due to the fact that as the mean $\theta_{[k]}$ increases in relation to $\theta_{[i]}$ $(1 \leqslant i \leqslant k-1)$ so does the corresponding variance $\alpha\theta^2_{[k]}$ and therefore the $P(\text{CS})$ remains bounded away from 1. We point out that this result corresponds to the discrepancy

between the large sample and the small sample results for the procedures based on T_{MMSS} that we discussed in Section 3; see Section 4.2.1 for additional details.

It may also be noted that for fixed k, δ^* and P^*, the sample size n required to guarantee (2.1) increases with α. As $\alpha \to \infty$, the required sample size n tends to the smallest integer $\geqslant \lambda^2(k, \delta^*, P^*)/2$.

4.1.2. *Tables for implementing* $\mathscr{P}_I$. The tables of $\lambda(k, \delta^*, P^*)$ for $k = 2(1)10$, $\delta^* = 1.2(0.2)3.0$ and $P^* = 0.99$, 0.975, 0.95, 0.90 and 0.75 are given at the end of the paper.

It may be noted that the r.h.s. of (4.2) equals

$$P\left\{Z_i \leqslant \frac{(\delta^*-1)\lambda}{(\delta^{*2}+1)^{\frac{1}{2}}} \ (1 \leqslant i \leqslant k)\right\},$$

where the Z_i are standard normal random variables with corr $(Z_i, Z_j) = \rho = \delta^{*2}/(\delta^{*2}+1)$ $(1 \leqslant i, j \leqslant k, i \neq j)$. The equicoordinate upper $100(1-P^*)$-percentage points of this multivariate normal distribution have been tabulated by Gupta, Nagel and Panchapakesan (1973) (they denote the percentage point by H) for $k = 2(1)11(2)51$, $P^* = 0.99$, 0.975, 0.95, 0.90, 0.75 and for selected values of ρ. Therefore for certain δ^*- and k-values not tabulated by us, one can use their tables as follows: First compute $\rho = \delta^{*2}/(\delta^{*2}+1)$ and by referring to the appropriate table read the H-value (note that their N equals our $k-1$). Then $\lambda(k, \delta^*, P^*) = H(\delta^{*2}+1)^{\frac{1}{2}}/(\delta^*-1)$. In fact our λ-entries for $\delta^* = 2.0$ and 3.0 have been obtained from their tables in the above manner. The remaining entries were computed on Northwestern University's CDC 6400 computer by solving the integral equation obtained by setting the r.h.s. of (4.2) equal to P^* using the IMSL subroutine ZSYSTM; the integral was evaluated using the Romberg method of numerical integration. The accuracy of the calculations was checked by computing some selected entries for $\delta^* = 2.0$ and $\delta^* = 3.0$ and comparing them with the corresponding entries in the Gupta *et al.* tables. The entries given in our tables are rounded off in the fourth decimal place and should not be off by more than one digit in the last decimal place reported. We remark that, with the help of interpolation, the tables in Milton (1963) can also be used to find the n-values for certain δ^*-, k-, and P^*-values.

We finally note that for P^*-values close to 1, an excellent *approximation to the sample size* can be obtained by using the result that as $P^* \to 1$

$$1-P^* \cong \frac{1}{\lambda}\binom{k-1}{\delta^*-1}\left(\frac{\delta^{*2}+1}{2\pi}\right)^{\frac{1}{2}} \exp\left[-(\delta^*-1)^2\lambda^2/(\delta^{*2}+1)\right]$$

where $\lambda = \lambda(k, \delta^*, P^*)$ and hence

$$\lambda^2 \cong \frac{(\delta^{*2}+1)}{(\delta^*-1)^2} \log_e \left(\frac{k-1}{1-P^*}\right). \quad \ldots (4.5)$$

The proof of (4.5) follows along similar lines as that of Lemma 6.2.1 of Bechhofer, Kiefer and Sobel (1968) and is hence omitted.

4.1.3. *Adequacy of the large sample approximation.* In this section we address the problem of how large should n be or equivalently how close should P^* be to 1 in order that the asymptotic approximation (4.1) for P_{LF} (CS) is valid. To this end we carried out a simulation study for procedure $\mathscr{P}_I$ employing the estimator of our choice T_{MMSS} for $k = 3$, $\delta^* = 1.2$, $\alpha = 0.5$ and $P^* = 0.75, 0.90, 0.95, 0.975, 0.99$. (A parallel study was carried out for $\mathscr{P}_I$ employing the estimator T^+_{LMMS} but the results were quite similar and hence are not reported here.)

The details of the simulation study are as follows. For each P^*, 1000 experiments were run. In each experiment $k = 3$ independent pairs of r.v.'s $\bar{X}_i \sim N(\theta_i, \alpha\,\theta_i^2/n)$ and $S_i^2 \sim \alpha\,\theta_i^2\,\chi^2_{(n-1)}/(n-1)$ where generated (by applying the standard transformations to uniform [0, 1] r.v.'s generated by the fortran library function RANF) where $\theta_1 = \theta_2 = 1$ and $\theta_3 = \delta^*$; n was taken to be the integer closest to $\alpha\lambda^2(k, \delta^*, P^*)/(1+2\alpha)$. The value of the estimator T_{MMSS} was then computed for each π_i from $\bar{X}_i$ and S_i^2 $(1 \leqslant i \leqslant 3)$ using the formula given in Section 3. Then by applying $\mathscr{P}_I$ it was determined whether π_3 (the best population) is correctly selected or not. The estimate of P_{LF} (CS) is given by the fraction of the experiments in which a correct selection is made. The results of the simulation study are given in Table 4.1 below.

TABLE 4.1. ESTIMATES OF P_{LF}(CS) FOR $\mathscr{P}_I$ EMPLOYING T_{MMSS} ($k = 3$, $\delta^* = 1.2$, $\alpha = 0.5$, no. of expts. = 1000)

P^*	n	P_{LF}(CS)	95% confidence limits for P_{LF}(CS)
0.75	15	0.730	(0.7025, 0.7575)
0.90	37	0.894	(0.8749, 0.9131)
0.95	55	0.939	(0.9242, 0.9538)
0.975	74	0.971	(0.9606, 0.9814)
0.99	99	0.988	(0.9813, 0.9947)

A study of this table indicates that in all the cases the actual $P_{LF}(\mathrm{CS})$ is not significantly different at 5% level from the specified P^*. Thus even for P^* as low as 0.75 the asymptotic approximation is quite good. However the approximation appears to be consistently deficient (estimated $P_{LF}(\mathrm{CS})$ is less than the specified P^*) although the extent of deficiency is rather small. Furthermore the deficiency appears to lessen as P^* increases.

4.2. *Procedure for Goal II (subset selection approach).* We propose the following single-stage procedure $\mathscr{P}_{II}$ for Goal II : Take $n \geqslant 2$ independent observations from each π_i, compute a B.A.N. estimator T_i of $\theta_i (1 \leqslant i \leqslant k)$ having the property of being a.s. nonnegative (T^+_{LMMS} and T_{MMSS} are two such estimators) and choose the subset of populations using the following rule : Include π_i in the subset $\Longleftrightarrow T_i \geqslant c^{-1} T_{\max} (1 \leqslant i \leqslant k)$ where $c (1 \leqslant c < \infty)$ is to be chosen so that (2.2) is guaranteed. It should be evident to the reader that if the T_i are not restricted to being a.s. nonnegative then the selected subset could be empty using the above rule.

4.2.1. *Probability of correct selection of $\mathscr{P}_{II}$ and its infimum.* To find the minimum value of c necessary to guarantee (2.2) using $\mathscr{P}_{II}$ we first derive the asymptotic expression for $P_{\boldsymbol{\theta}}(\mathrm{CS})$ of $\mathscr{P}_{II}$ and then find its infimum over Ω. This is done in the following theorem.

Theorem 4.2 : *For $\mathscr{P}_{II}$ we have that for any $\boldsymbol{\theta} \in \Omega$*

$$P_{\boldsymbol{\theta}}(CS) = \int_{-\infty}^{\infty} \prod_{i=1}^{k-1} \Phi[t\delta_{ki}c + (\delta_{ki}c - 1)\lambda]\, \varphi(t)\, dt, \qquad \ldots (4.6)$$

where $\lambda = \{n(1+2\alpha)/\alpha\}^{\frac{1}{2}}$. The infimum of $P_{\boldsymbol{\theta}}(CS)$ over Ω is achieved for any $\boldsymbol{\theta}$ such that $\delta_{ki} = 1 \forall\ i \neq k$ (known as the equal means or EM-configuration) and

$$\inf_{\Omega} P_{\boldsymbol{\theta}}(CS) = \int_{-\infty}^{\infty} \Phi^{k-1}[tc + (c-1)\lambda]\, \varphi(t)\, dt. \qquad \ldots (4.7)$$

Proof : The proof is exactly analogous to that of Theorem 4.1 and is hence omitted.

Let $c = c(k, \lambda, P^*)$ denote the solution in c to the equation obtained by setting the r.h.s. of (4.7) equal to P^*. Then the probability requirement (2.2) can be guaranteed by using that solution in procedure $\mathscr{P}_{II}$. For fixed $\lambda > 0$, the r.h.s of (4.7) is strictly increasing in c for $c > 1$. As $c \to 1$, the r.h.s. $\to 1/k$ and as $c \to \infty$, the r.h.s. $\to \Phi(\lambda)$. Therefore a unique solution in c exists for each P^*-value, $1/k < P^* < \Phi(\lambda)$. This implies the nonexistence

of $\mathscr{P}_{II}$ for $P^* \geqslant \Phi(\lambda)$ which we had earlier pointed out in Section 3. We might note however that, for all practical purposes, if $n \geqslant (3.5)^2/2 \geqslant (3.5)^2\alpha/(1+2\alpha)$ then this upper bound is extremely close to 1. Therefore, $\mathscr{P}_{II}$ can be used to guarantee (2.2) for P^*-values usually encountered in practice.

4.2.2. *Tables for implementing* $\mathscr{P}_{II}$. To obtain the values of c, the tables of λ prepared for Goal I can be used in an approximate fashion as follows : First compute the value of $\lambda = \{n(1+2\alpha)/\alpha\}^{\frac{1}{2}}$. In the table for the given value of k, enter the row corresponding to the specified value of P^* and go down the row until two adjacent λ-entries are found which straddle the computed value of λ. Read off the corresponding adjacent δ^*-values. Now note that δ^* plays the role of c and therefore the appropriate value of c can be found by interpolation between the adjacent δ^*-values.

It might be useful to prepare separate tables for the values of $c = c(k, \lambda, P^*)$ but we have not attempted to do so in the present paper.

4.2.3. *Some properties of* $\mathscr{P}_{II}$: (a) *Expected size of the selected subset.* A measure of performance of $\mathscr{P}_{II}$ is the associated expected size of the subset say, $E_{\boldsymbol{\theta}}(S)$. We have the following asymptotic expression

$$E_{\boldsymbol{\theta}}(S) \cong \sum_{i=1}^{k} \int_{-\infty}^{\infty} \prod_{\substack{j=1 \\ j \neq i}}^{k-1} \Phi[t\delta_{ij}c + (\delta_{ij}c - 1)\lambda]\, \varphi(t)\, dt. \qquad \ldots (4.8)$$

It is often of interest to identify the parameter configuration where the supremum of (4.8) occurs. For $k = 2$, denoting $\delta_{21} = \delta$ it is easy to see that (4.8) becomes

$$E_{\boldsymbol{\theta}}(S) \cong \Phi\left[\frac{(\delta c - 1)\lambda}{(\delta^2 c^2 + 1)^{\frac{1}{2}}}\right] + \Phi\left[\frac{(c-\delta)\lambda}{(c^2+\delta^2)^{\frac{1}{2}}}\right] \qquad \ldots (4.9)$$

The supremum of (4.9) occurs at $\delta = 1$, i.e., the EM-configuration, and this supremum equals $2P^*$. For $k > 2$ it appears difficult to obtain such a result because of the lack of the MLR property of $N(\theta, \alpha\theta^2)$ distribution. We can only state that $\sup_{\Omega} E_{\boldsymbol{\theta}}(S) \geqslant E_{EM}(S) = kP^*$.

(b) *Monotonicity property of* $\mathscr{P}_{II}$. If p_i denotes the probability that π_i is included in the subset using $\mathscr{P}_{II} (1 \leqslant i \leqslant k)$ then we can show using the method of the proof of Theorem 4.1 that

$$\theta_i \geqslant \theta_j \Longrightarrow p_i \geqslant p_j. \qquad \ldots (4.10)$$

5. Concluding remarks

In this paper we have provided new ranking and selection procedures for the normal means problem when the coefficient of variation is the same for all populations and is known. When such a situation obtains in practice and suppose, e.g., that the experimenter is interested in Goal I, then our procedure $\mathscr{P}_I$ should be used instead of the two-stage procedure of Dudewicz and Dalal for unknown and unequal variances. We should, however, point out that any comparison between their procedure and our procedure is precluded because different distance functions are used in the two procedures.

For future research, it would be useful to extend these procedures to the case where the common coefficient of variation is unknown.

Appendix

Small sample results

A.1. *Distribution of* T^+_{LMMS}. Consider the set-up of Section 3. We shall derive the exact small sample results only for T^+_{LMMS}; it appears impossible to obtain the corresponding results for T_{MMSS}. In the following we shall denote T^+_{LMMS} by T for convenience.

By a simple conditioning argument on S/θ it is easy to see that the c.d.f. of T/θ is given by

$$F_n(t) = \begin{cases} 0 & \text{for } t < 0 \\ \int_0^\infty \Phi\left[\frac{\sqrt{n}}{d_n}(t - d_n - c_n u)\right] h_{n-1}(u)\, du & \text{for } t \geqslant 0, \end{cases} \quad \ldots \quad (A.1)$$

where $h_\nu(\cdot)$ denotes the p.d.f of a $(\chi^2_\nu/\nu)^{\frac{1}{2}}$ r.v.

A.2. *Procedure for Goal I (indifference-zone approach).* In the following theorem we derive an exact expression for $P_{\boldsymbol{\theta}}(\mathrm{CS})$ of procedure $\mathscr{P}_I$ described in Section 4.1 and find its infimum over $\Omega(\delta^*)$.

Theorem A.1 : *For* $\mathscr{P}_I$ *we have that for any* $\boldsymbol{\theta} \in \Omega$,

$$P_{\boldsymbol{\theta}}(CS) = \frac{1}{k}[F_n(0)]^k + \sum_{s \in \mathscr{S}} [F_n(0)]^{k-1-|s|} \times \int_{0+}^{\infty} \prod_{i \in s} [F_n(t\delta_{ki}) - F_n(0)]\, dF_n(t), \quad \ldots \quad (A.2)$$

where $\mathcal{S}$ = *collection of all subsets from* $\{1, 2, \ldots, k-1\}$ *and* $|s|$ = *cardinality of the set* s. *The infimum of* $P_{\boldsymbol{\theta}}(CS)$ *over* $\Omega(\delta^*)$ *is achieved at any* $\boldsymbol{\theta}$ *such that* $\delta_{ki} = \delta^*$ *for* $\forall\ i \neq k$ (*LF-configuration*) *and*

$$\inf_{\Omega(\delta^*)} P_{\boldsymbol{\theta}}(CS) = \frac{1}{k}[F_n(0)]^k + \int_{0+}^{\infty} [F_n(t\delta^*)]^{k-1} dF_n(t). \qquad \ldots \quad (A.3)$$

Proof: We have that

$$P_{\boldsymbol{\theta}}(\mathrm{CS}) = \frac{1}{k} P_{\boldsymbol{\theta}}\{T_{(i)} = 0\ \forall\ i \neq k\} + \sum_{s \in \mathcal{S}} P_{\boldsymbol{\theta}}\{T_{(ik)} > T_{(i)}\ \forall\ i \neq k,$$

$$T_{(i)} > 0\ \forall\ i \in s,\ T_{(i)} = 0\ \forall\ i \notin s\}$$

$$= \frac{1}{k}[F_n(0)]^k + \sum_{s \in \mathcal{S}} [F_n(0)]^{k-1-|s|}$$

$$\times P_{\boldsymbol{\theta}}\{(T_{(k)}/\theta_{[k]})\delta_{ki} > T_{(i)}/\theta_{[i]} > 0\ \forall\ i \in s\}$$

which yields (A.2). The proof of the LF-configuration is obvious from the monotonicity in δ_{ki} of (A.2). The expression (A.3) then follows immediately.

The probability requirement (2.1) can now be guaranteed by finding the smallest n which would make the r.h.s. of (A.3) $\geqslant P^*$ and using this n in $\mathscr{P}_I$. It can be checked that as $n \to \infty$ the r.h.s. of (A.3) $\to 1$. However it is not clear whether it is monotonic in n or not. In view of the computational and other difficulties cited in Section 2 we have not attempted tabulation of the exact n-values for selected values of α, k, δ^*, P^*.

It may be noted that for the generalized LF-configuration (i.e., $\delta_{ki} = \delta > 1\ \forall\ i \neq k$) for fixed $n \geqslant 2$, if $\delta \to 1$ then $P_{\boldsymbol{\theta}}(\mathrm{CS}) \to 1/k$ and if $\delta \to \infty$ then $P_{\boldsymbol{\theta}}(\mathrm{CS}) \to 1 - F_n(0) - [F_n(0)]^k/k < 1$. These results correspond to the similar ones obtained in the case of large samples.

A.3. *Procedure for Goal* II (*subset selection approach*). In the following theorem we derive an exact expression for $P_{\boldsymbol{\theta}}(\mathrm{CS})$ of $\mathscr{P}_{II}$ described in Section 4.2 and find its infimum over Ω.

Theorem A.2 : *For* $\mathscr{P}_{II}$ *we have that for any* $\boldsymbol{\theta} \in \Omega$

$$P_{\boldsymbol{\theta}}(CS) = [F_n(0)]^k + \sum_{s \in \mathcal{S}} [F_n(0)]^{k-1-|s|}$$

$$\times \int_{0+}^{\infty} \prod_{i \in s} [F_n(t\delta_{ki}c) - F_n(0)]\, dF_n(t). \qquad \ldots \quad (A.4)$$

The infimum of $P_{\boldsymbol{\theta}}(\mathrm{CS})$ over Ω is achieved at any $\boldsymbol{\theta}$ such that $\delta_{kl} = 1 \; \forall \; i \neq k$ (EM-configuration) and

$$\inf_{\Omega} P_{\boldsymbol{\theta}}(\mathrm{CS}) = [F_n(0)]^k + \int_{0+}^{\infty} [F_n(tc)]^{k-1} dF_n(t). \qquad \ldots \quad (A.5)$$

Proof: The proof is similar to that of Theorem A.1 and is hence omitted.

Let $c = c(n, \alpha, k, P^*)$ denote the solution in c to the equation obtained by setting the r.h.s. of (A.5) equal to P^*. Then the probability requirement (2.2) can be guaranteed by using that value of c in procedure $\mathscr{P}_{II}$. The r.h.s. of (A.5) is strictly increasing in c for $c > 1$. As $c \to 1$, the r.h.s. $\to 1/k$ and as $c \to \infty$, the r.h.s. $\to [F_n(0)]^k + 1 - F_n(0) < 1$. Therefore a unique solution in c exists for each P^*, $1/k < P^* < [F_n(0)]^k + 1 - F_n(0)$. This implies that $\mathscr{P}_{II}$ does not exist for $P^* \geqslant [F_n(0)]^k + 1 - F_n(0)$ which corresponds to the similar result obtained in the case of large samples.

We omit the details concerning the exact small sample properties of $\mathscr{P}_{II}$ for the sake of brevity.

TABLES. VALUES OF $\lambda(k, \delta^*, P^*)$ FOR $k = 2(1)\ 10$, $\delta^* = 1.2(0.2)\ 3.0$, AND $P^* = 0.99, 0.975, 0.95, 0.90, 0.75$

					$k = 2$					
P^* \ δ^*	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0
0.99	18.1667	10.1947	7.5127	5.9869	5.2018	4.6842	4.3197	4.0497	3.8421	3.6782
0.975	15.3081	8.5905	6.3306	5.0449	3.8416	3.9471	3.6400	3.4125	3.2375	3.0990
0.95	12.8479	7.2099	5.3132	4.2341	3.6781	3.3128	3.0550	2.8640	2.7172	2.6008
0.90	10.0127	5.6189	4.1407	3.2997	2.8666	2.5817	2.3808	2.2320	2.1176	2.0264
0.75	5.2641	2.9541	2.1769	1.7348	1.5082	1.3573	1.2517	1.1735	1.1133	1.0665
					$k = 3$					
0.99	19.9068	10.9203	7.9521	6.4830	5.6105	5.0340	4.6267	4.3238	4.0900	3.9045
0.975	17.1843	9.4100	6.8410	5.5689	4.8129	4.3138	3.9603	3.6972	3.4944	3.3332
0.95	14.8514	8.1145	5.8871	4.7837	4.1278	3.6944	3.3873	3.1589	2.9824	2.8423
0.90	12.1710	6.6245	4.7897	3.8799	3.3387	2.9808	2.7271	2.5383	2.3923	2.2764
0.75	7.7136	4.1434	2.9601	2.3723	2.0219	1.7897	1.6249	1.5020	1.4068	1.3310

$k = 4$

P^* \ δ^*	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0
0.99	20.8557	11.4169	8.2958	6.7490	5.8290	5.2206	4.7890	4.4682	4.4202	4.0230
0.975	18.1978	9.9364	7.2031	5.8476	5.0408	4.5072	4.1287	3.8467	3.6285	3.4553
0.95	15.9222	8.6675	6.2657	5.0736	4.3639	3.8942	3.5608	3.3124	3.1202	2.9673
0.90	13.3117	7.2091	5.1873	4.1831	3.5846	3.1881	2.9066	2.6966	2.5342	2.4048
0.75	8.9757	4.7827	2.3909	2.6982	2.2846	2.0099	1.8146	1.6685	1.5549	1.4654

$k = 5$

0.99	21.5012	11.7535	8.5280	6.9281	5.9757	5.3453	4.8981	4.5644	4.3068	4.1019
0.975	18.8828	10.2911	7.4463	6.0343	5.1930	4.6361	4.2407	3.9459	3.7178	3.5362
0.95	16.6425	9.0380	6.5185	5.2668	4.5211	4.0268	3.6758	3.4140	3.2113	3.0499
0.90	14.0730	7.5982	5.4513	4.3841	3.7474	3.3251	3.0250	2.8010	2.6275	2.4893
0.75	9.8080	5.2036	3.6741	2.9122	2.4568	2.1542	1.9388	1.7777	1.6529	1.5532

$k = 6$

0.99	21.9843	12.0061	8.7019	7.0619	6.0850	5.4380	4.9788	4.6358	4.3711	4.1600
0.975	19.3960	10.5561	7.6277	6.1733	5.3062	4.7318	4.3238	4.0193	3.7837	3.5960
0.95	17.1799	9.3139	6.7064	5.4105	4.6376	4.1250	3.7608	3.4890	3.2785	3.1107
0.90	14.6384	7.8867	5.6469	4.5327	3.8675	3.4262	3.1124	2.8779	2.6963	2.5515
0.75	10.4214	5.5135	3.8824	3.0695	2.5833	2.2601	2.0300	1.8578	1.7242	1.6175

$k = 7$

0.99	22.3735	12.2072	8.8400	7.1681	6.1715	5.5115	5.0427	4.6929	4.4218	4.2065
0.975	19.8041	10.7665	7.7715	6.2832	5.3956	4.8073	4.3893	4.0772	3.8356	3.6429
0.95	17.6060	9.5320	6.8550	5.5238	4.7293	4.2023	3.8278	3.5481	3.3313	3.1586
0.90	15.0854	8.1145	5.8010	4.6497	3.9621	3.5057	3.1810	2.9383	2.7503	2.6003
0.75	10.9035	5.7569	4.0459	3.1928	2.6824	2.3431	2.1014	1.9205	1.7800	1.6679

$k = 8$

0.99	22.6955	12.3737	8.9543	7.2557	6.2386	5.5720	5.0952	4.7394	4.4634	4.2444
0.975	20.1414	10.9403	7.8901	6.3739	5.4694	4.8695	4.4431	4.1248	3.8782	3.6817
0.95	17.9573	9.7122	6.9769	5.6166	4.8049	4.2659	3.8827	3.5965	3.3747	3.1979
0.90	15.4531	8.3018	5.9277	4.7458	4.0397	3.5709	3.2372	2.9878	2.7945	2.6403
0.75	11.2986	5.9561	4.1797	3.2938	2.7636	2.4110	2.1597	1.9717	1.8257	1.7092

$k = 9$

P^* \ δ^*	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0
0.99	22.9695	12.5154	9.0514	7.3302	6.3037	5.6233	5.1397	4.7788	4.4987	4.2763
0.975	20.4287	11.0880	7.9907	6.4507	5.5318	4.9221	4.4887	4.1650	3.9142	3.7143
0.95	18.2562	9.8649	7.0806	5.6955	4.8686	4.3196	3.9292	3.6375	3.4113	3.2309
0.90	15.7650	8.4603	6.0348	4.8270	4.1052	3.6259	3.2847	3.0296	2.8318	2.6740
0.75	11.6321	6.1243	4.2925	3.3789	2.8320	2.4682	2.2089	2.0148	1.8642	1.7438

$k = 10$

0.99	23.2076	12.6385	9.1356	7.3947	6.3562	5.6677	5.1783	4.8128	4.5292	4.3042
0.975	20.6779	11.2160	8.0779	6.5172	5.5857	4.9676	4.5281	4.1998	3.9453	3.7424
0.95	18.5147	9.9972	7.1703	5.7637	4.9238	4.3661	3.9693	3.6728	3.4429	3.2595
0.90	16.0346	8.5974	6.1274	4.8972	4.1620	3.6734	3.3257	3.0657	2.8639	2.7031
0.75	11.9199	6.2694	4.3898	3.4522	2.8910	2.5175	2.2513	2.0520	1.8973	1.7737

Acknowledgements

The author wishes to thank Professor Jerome Sacks for some helpful discussions and the referee for his comments which resulted in Section 4.1.3. The revision of this paper was supported by NSF Grant No. ENG77-06112.

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Paper received : July, 1977.

Revised : June, 1978.